THE COMPLETE INTEGRABILITY OF A LIE-POISSON SYSTEM PROPOSED BY BLOCH AND ISERLES

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Dedicated with affect and admiration to Percy Deift on the occasion of his 60th birthday

Abstract. We establish the Liouville integrability of the differential equation $\dot{S}(t) = [N, S^2(t)]$, recently considered by Bloch and Iserles. Here, N is a real, fixed, skew-symmetric matrix and S is real symmetric. The equation is realized as a Hamiltonian vector field on a coadjoint orbit of a loop group, and sufficiently many commuting integrals are presented, together with a solution formula for their related flows in terms of a Riemann-Hilbert factorization problem. We also answer a question raised by Bloch and Iserles, by realizing the same system on a coadjoint orbit of a finite dimensional Lie group.

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1. Introduction.

In a recent paper [BI], Bloch and Iserles considered the differential equation

$$\dot{S}(t) = [N, S^2(t)], \qquad (*)$$

where both S and N are real $n \times n$ matrices, S is symmetric and N is a constant skew-symmetric matrix. The similarity with the celebrated Toda differential equation ([F], [Mo]) is hard to miss, and one might expect that both equations have some common properties. Indeed, from the equivalent Lax pair formulation

$$\dot{S} = [NS + SN, S]$$

Bloch and Iserles deduced that S evolves by orthogonal conjugation: this implies global existence of solutions and spectrum invariance of $S(t)$. Also, following Manakov's approach to the rigid body equation [Ma], they showed that (*) is equivalent to the following Lax equation with spectral parameter

$$(S + zN)^{\cdot} = [NS + SN + zN^2, S + zN],$$

from which a large collection of conserved quantities is clear, namely, the nontrivial coefficients in z in the expansions of $tr(S + zN)^k$. Even more, they showed that equation (*) is a Lie-Poisson system on an appropriate phase space and, corroborated by numerical experiments, conjectured the complete integrability of the system.

In this paper, we shall formulate equation (*) as a Hamiltonian system on a coadjoint orbit of a loop group, in the spirit of [DLT2]. The situation is rather familiar: most of the difficulty lies in guessing the loop group and a convenient representation of the dual of its Lie algebra. In this case, the group is LG_-^Σ, the set of smooth loops $g : \mathbb{S}^1 \to GL(N, \mathbb{C})$ satisfying the reality condition $\overline{g(z)} = g(\bar{z})$ which admit an analytic continuation to the exterior of the unit circle with $g(\infty) = I$, the identity matrix, and which are kept fixed by the involution $\Sigma(g)(z) = (g(-z)^T)^{-1}$. The nondegenerate pairing on arbitrary loops

$$(X, Y) = \oint_{|z|=1} tr(X(z)Y(z)) \frac{dz}{2\pi i} = \sum_j tr(X_j Y_{-j-1})$$

identifies the Lie algebra dual $(L\mathfrak{g}_-^\sigma)^*$ with the set of power series $S_0 + N_1 z + \ldots$, where the coefficients of the powers $z^k, k = 0, 1, \ldots$, are alternatively symmetric and skew-symmetric real matrices.

The proof of Liouville integrability of the Bloch-Iserles equation (*) takes two familiar forms. In one of them, embed LG_-^Σ into the larger loop group LG^Σ of all smooth invertible loops and there search for a second bracket structure, induced by an R-matrix, from which the existence of commuting integrals follows. Otherwise, consider the Lie group $LG_+^\Sigma \times LG_-^\Sigma$, corresponding to the Lie algebra anti-direct sum $L\mathfrak{g}_+^\sigma \ominus L\mathfrak{g}_-^\sigma$ which as a manifold can be identified with the subset of loops in LG^Σ which admit a Birkhoff factorization $g = g_+ g_-^{-1}$ with trivial diagonal part ([PS] is a good reference for the few facts about Birkhoff factorization which will be used in this paper). The Lie bracket of $L\mathfrak{g}_+^\sigma \ominus L\mathfrak{g}_-^\sigma$ is of course identical to the bracket defined by the R-matrix above. Both points of view will be described in an intertwined fashion in Sections 2 and 3. Integrability is then proved in Section 4 by showing that the integrals suggested by Bloch and Iserles are indeed commuting, and generically span half of the dimension of the coadjoint orbit of interest. A formula for the commuting flows in terms of a Birkhoff factorization is the content of Section 5.

The loop group scenario embeds equation (*) for $n \times n$ matrices into a coadjoint orbit $\mathcal{O}$ of generic dimension $2[n^2/4]$ (here $[x]$ is the greatest integer less than or equal to x). In Section 6, we consider a question raised by Bloch and Iserles in their original paper: to find a finite dimensional matrix group such that their equation is realized as a Hamiltonian system on the dual of its Lie algebra. In [BI], the authors presented an interesting algorithm to study this problem, adapted from a constructive proof of Ado's theorem, which states that any finite dimensional Lie algebra admits a matricial faithful representation. From the loop group setup, we obtain an explicit realization: there is an n^2-dimensional nilpotent matrix group G_f, faithfully realized on $3n \times 3n$ matrices, giving rise to a coadjoint orbit isomorphic

to $\mathcal{O}$. The prescription yielding the finite dimensional realization is simple and rather general. Since the orbit $\mathcal{O}$ is described in terms of (bi-)truncated Laurent expansions of its elements, one should expect that the full group is not needed to cover the orbit. Indeed, as we shall see, only the first three terms of a loop in LG_-^Σ are relevant to the computation of the coadjoint action on the orbit of interest. This suggests taking a quotient $G_f = LG_-^\Sigma / LG_{-3}^\Sigma$ by the normal subgroup LG_{-3}^Σ of loops of the form $I + O(z^{-3})$: it turns out that G_f admits a simple Heisenberg-like representation.

In the final section, we present an example of an infinite dimensional orbit obtained by the same coadjoint action. Here, the Hamiltonian which gives rise to equation (*) in the small orbit now induces a system of partial differential equations containing some simple integral terms.

Arieh Iserles informed us that a forthcoming paper, together with A. Bloch, T. Ratiu and J. Marsden, will be dedicated to a different proof of the integrability of the Bloch-Iserles equation, using very different techniques.

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2. The Lie algebraic setup and the Poisson structure.

Let G be $GL(N,\mathbb{R})$ with Lie algebra $\mathfrak{g}$, and let LG be the group of loops $g : S^1 \longrightarrow GL(N,\mathbb{C})$ satisfying the reality condition $\overline{g(z)} = g(\bar{z})$. We consider the involutive automorphism $\Sigma : LG \longrightarrow LG$, given by $\Sigma(g)(z) = (g(-z)^T)^{-1}$ and its fixed point set (the stable locus)

$$LG^\Sigma = \{\, g \in LG \mid g(z)(g(-z))^T = I \,\}. \tag{2.1}$$

Clearly, LG^Σ is a Lie subgroup of LG. We shall denote by LG_+^Σ (resp. LG_-^Σ) the Lie subgroup of LG^Σ consisting of loops which extend analytically to the interior (resp. to the exterior) of the unit circle, with the additional requirement that loops in LG_-^Σ take the value I, the identity matrix, at ∞.

Let $L\mathfrak{g}$ be the Lie algebra of LG, consisting of loops $X(z) = \sum_{-\infty}^{\infty} X_j z^j$ with coefficients $X_j \in \mathfrak{g}$. Then the Lie algebra $L\mathfrak{g}^\sigma$ of LG^Σ is the stable locus of the linearization $\sigma : L\mathfrak{g} \longrightarrow L\mathfrak{g}$ of Σ, given by the formula

$$\sigma(X)(z) = -(X(-z))^T = \sum_j \theta(X_j)(-z)^j \tag{2.2}$$

where θ is the Cartan involution $\xi \mapsto -\xi^T$. Therefore, if $\mathfrak{k}$ (resp. $\mathfrak{p}$) denote the $+1$ (resp. -1) eigenspace of θ, consisting of skew-symmetric (resp. symmetric) matrices, then explicitly,

$$L\mathfrak{g}^\sigma = \{\, X \in L\mathfrak{g} \mid X_{2j} \in \mathfrak{k},\ X_{2j+1} \in \mathfrak{p} \ \text{ for all } j \,\}. \tag{2.3}$$

From the definition of LG_+^Σ (resp. LG_-^Σ), it is clear that its Lie algebra $L\mathfrak{g}_+^\sigma$ (resp. $L\mathfrak{g}_-^\sigma$) consists of elements of the form $\sum_{j\ge 0} X_j z^j$ (resp. $\sum_{j<0} X_j z^j$). Hence the elements in $L\mathfrak{g}_-^\sigma$ are equal to 0 at ∞ and we have the splitting

$$L\mathfrak{g}^\sigma = L\mathfrak{g}_+^\sigma \oplus L\mathfrak{g}_-^\sigma \tag{2.4}$$

with associated projection maps Π_+ and Π_-.

We now introduce

$$\widetilde{G} = \{g \in LG^{\Sigma} \mid g = g_+ g_-^{-1}, \text{ where } g_- \in LG_-^{\Sigma},\ g_+ \in LG_+^{\Sigma}\}. \tag{2.5}$$

Then an easy argument shows that for $g \in \widetilde{G}$, the factors $g_\pm$ are unique (more details will be provided in Section 5). Moreover, from the Birkhoff factorization theorem [PS], $\widetilde{G}$ is a dense open subset of LG^{Σ} in the natural topology. Following the procedure in [DLT1],[DLT2], we can endow $\widetilde{G}$ with a Lie group structure by defining the multiplication

$$g * h \equiv g_+ h g_-^{-1}. \tag{2.6}$$

Clearly, the map $\Psi : \widetilde{G} \longrightarrow LG_+^{\Sigma} \times LG_-^{\Sigma}$ given by $\Psi(g) = \Psi(g_+ g_-^{-1}) \mapsto (g_+, g_-)$ is a Lie group isomorphism, when the image is equipped with the product group structure. Consequently, the pull-back of the standard Lie bracket on $L\mathfrak{g}_+^{\sigma} \oplus L\mathfrak{g}_-^{\sigma}$ (Lie algebra direct sum) under $\psi = T_e\Psi : X \mapsto (\Pi_+ X, -\Pi_- X)$ yields the Lie bracket on $\widetilde{\mathfrak{g}} = Lie(\widetilde{G})$:

$$[X, Y]_{\widetilde{\mathfrak{g}}} = [\,\Pi_+ X, \Pi_+ Y\,] - [\,\Pi_- X, \Pi_- Y\,] \tag{2.7}$$

for all $X, Y \in \widetilde{\mathfrak{g}}$. Similarly, one may use ψ to obtain a formula for the adjoint action of $\widetilde{G}$ on $\widetilde{\mathfrak{g}}$,

$$Ad_{\widetilde{G}}(g)X = g_-(\Pi_- X)g_-^{-1} + g_+(\Pi_+ X)g_+^{-1}. \tag{2.8}$$

Remark 2.1 By standard r-matrix theory [STS],

$$R = \Pi_+ - \Pi_- \tag{2.9}$$

is a solution of the modified Yang-Baxter equation, i.e.,

$$[RX, RY] - R([RX, Y] + [X, RY]) = -[X, Y] \tag{2.10}$$

for all $X, Y \in L\mathfrak{g}^{\sigma}$. By an easy computation, we can show that the R-bracket on $L\mathfrak{g}^{\sigma}$ given by

$$[X, Y]_R = \frac{1}{2}([RX, Y] + [X, RY]) \tag{2.11}$$

coincides with $[\cdot,\cdot]_{\widetilde{\mathfrak{g}}}$. Thus the vector space $L\mathfrak{g}^{\sigma}$ equipped with the Lie bracket $[\cdot,\cdot]_R$ is identical to the Lie algebra $\widetilde{\mathfrak{g}}$.

To identify the duals of the Lie algebras, we introduce the following nondegenerate invariant pairing on $L\mathfrak{g}$:

$$\begin{aligned}(X, Y) &= \oint_{|z|=1} tr(X(z)Y(z))\,\frac{dz}{2\pi i} \\ &= \sum_j tr(X_j Y_{-j-1}).\end{aligned} \tag{2.12}$$

As the reader will see, this choice of pairing is critical for what we have in mind. Using this pairing, we identify the algebraic dual $(L\mathfrak{g}^{\sigma})^*$ of $L\mathfrak{g}^{\sigma}$ with the stable locus of the Lie algebra anti-isomorphism $\sigma^* = -\sigma$:

$$L\mathfrak{g}^{\sigma^*} = \{ X \in L\mathfrak{g} \mid X_{2j+1} \in \mathfrak{k},\ X_{2j} \in \mathfrak{p} \text{ for all } j \}. \tag{2.13}$$

Notice that the alternation of symmetric and skew-symmetric coefficients in the Laurent expansion still holds, now with a parity opposite to the one found in elements in the Lie algebra. By an easy computation making use of (2.8) and (2.12), we find

$$Ad^*_{\widetilde{G}}(g)A = \Pi_-(g_+^{-1}Ag_+) + \Pi_+(g_-^{-1}Ag_-) \tag{2.14}$$

where we have used the fact that $\Pi_+^* = \Pi_-$ and $\Pi_-^* = \Pi_+$.

Remark 2.2 In the general context of an involutive automorphism of a finite dimensional Lie algebra, the author in [R] also considered the stable locus of the corresponding involution on the loop algebra. However, a different choice of pairing was used in [R], thus leading to a different identification of the dual.

Finally, the Lie-Poisson structure for smooth functions on $\tilde{\mathfrak{g}}^*$ is given by the usual formula:

$$\{F_1, F_2\}(X) = (X, [dF_1(X), dF_2(X)]_{\widetilde{\mathfrak{g}}}). \tag{2.15}$$

3. A Hamiltonian for the Bloch-Iserles equation.

The following result is standard.

Proposition 3.1. *For $1 \le k \le n$, $\ell \in \mathbb{Z}$, define*

$$H_{k\ell}(X) = \frac{1}{(k+1)} \oint_{|z|=1} tr(X(z)^{k+1}) \ \frac{dz}{2\pi i z^{\ell+1}}, \tag{3.1}$$

then the $H_{k\ell}$'s Poisson commute with respect to $\{\cdot,\cdot\}$. The Hamiltonian equation of motion generated by $H_{k\ell}$ is given by

$$\dot{X}(z) = \left[\Pi_+((X(z)^k z^{-(\ell+1)}), X(z) \right]. \tag{3.2}$$

Proof. Since $Ad_g^*(X)(z) = g(z)^{-1}X(z)g(z)$, the Hamiltonians are invariant under the coadjoint action of LG^{Σ}, i.e.

$$H_{k\ell}(Ad_g^*(X)) = H_{k\ell}(X), \quad g \in LG^{\Sigma}.$$

By classical r-matrix theory, we then conclude that they Poisson commute. The equation of motion for $H_{k\ell}$ is a straightforward computation. $\square$

Our next task is to search for interesting finite dimensional coadjoint orbits in $\widetilde{\mathfrak{g}}^*$. For any $m, n \in \mathbb{Z}_+$, define

$$\widetilde{\mathfrak{g}}^*_{(m,n)} = \{ X \in L\mathfrak{g}^{\sigma^*} \mid X(z) = \sum_{j=-m}^{n} X_j z^j \}. \tag{3.3}$$

Clearly, we can extend this definition to the case where $m = 0$ in which case we set $\widetilde{\mathfrak{g}}^*_n = \widetilde{\mathfrak{g}}^*_{(0,n)}$.

Proposition 3.2. *The sets* $\widetilde{\mathfrak{g}}^*_{(m,n)}$ *are invariant under* $Ad^*_{\widetilde{G}}(g)$ *for any* $g \in \widetilde{G}$.

Proof. Take $X \in \widetilde{\mathfrak{g}}^*_{(m,n)}$. For $m = 0$, the analyticity property of $g_\pm$ implies that

$$\Pi_-(g_+^{-1} X g_+) = 0,$$

$$\Pi_+(g_-^{-1} X g_-) = X_n z^n + \sum_{j=0}^{n-1} B_j z^j$$

for some matrices B_j, and we are done (we have used $g_-(\infty) = I$ in deriving the second line above). For $m \in \mathbb{Z}_+$, note that

$$Ad^*_{\widetilde{G}}(g)X = \Pi_-(g_+^{-1}(\Pi_- X)g_+) + \Pi_+(g_-^{-1}(\Pi_+ X)g_-).$$

Therefore, the same calculation as before applied to $\Pi_+ X$ shows that

$$\Pi_+(g_-^{-1}(\Pi_+ X)g_-) = X_n z^n + \sum_{j=0}^{n-1} B_j z^j$$

for some matrices B_j. On the other hand, it is easy to check that

$$\Pi_-(g_+^{-1}(\Pi_- X)g_+) = g_+(0)^T X_{-m} g_+(0) + \sum_{j=-m+1}^{-1} C_j z^j$$

for some matrices C_j. Putting the above two expressions together, the assertion follows. □

As a consequence of this result, we obtain finite dimensional coadjoint orbits through the elements in $\widetilde{\mathfrak{g}}^*_n$ and $\widetilde{\mathfrak{g}}^*_{(m,n)}$ and the Hamiltonian equation in (3.2) restricts to these orbits. In the rest of the section, we shall focus on the case $\widetilde{\mathfrak{g}}^*_1$, in which the Bloch-Iserles equation lies.

Proposition 3.3. *(a) Consider the loop* $S_0 + zN_0 \in \widetilde{\mathfrak{g}}^*_1$. *The* $Ad^*_{\widetilde{G}}$*-orbit through* $S_0 + zN_0$ *is given by the affine linear space*

$$\mathcal{O}_{S_0+zN_0} = \{Ad^*_{\widetilde{G}}(g)(S_0 + zN_0) \mid g \in \widetilde{G}\} = \{(S_0 + [\,N_0, P]) + zN_0 \mid P \in \mathfrak{p}\}.$$

In particular, if N_0 *has simple spectrum, the tangent space of* $\mathcal{O}_{S_0+zN_0}$ *is naturally identified with the vector space of real, symmetric matrices which are orthogonal to the matrix polynomials* $p(N_0^2)$.
(b) The Hamiltonian equation of motion

$$(S + zN)^{\cdot} = \left[\,\Pi_+((S + zN)^2/z)), S + zN\,\right] \tag{3.5}$$

generated by H_{20} on the Poisson submanifold $\widetilde{\mathfrak{g}}_1^*$ is equivalent to the Bloch-Iserles equation

$$\begin{aligned} \dot{N} &= 0, \\ \dot{S} &= [\,NS + SN, S\,]. \end{aligned} \tag{3.6}$$

Proof. We have

$$g_-(z) = I + g_-'(\infty) z^{-1} + O(z^{-2})$$

where $g_-'(\infty) = \frac{dg_-}{dz^{-1}}(z = \infty)$ and a simple computation obtains

$$\mathcal{O}_{S_0 + zN_0} = \{S_0 + [\,N_0, g_-'(\infty)] + zN_0\}.$$

Since $g_-^{-1}(z) = (g_-(-z))^T$, the matrix $g_-'(\infty)$ must be symmetric and, conversely, any symmetric matrix $P \in \mathfrak{p}$ is $g_-'(\infty)$ for an appropriate loop g_-. Now, endow $\mathfrak{p}$ with the usual inner product $< P, Q >= \ tr\ PQ$. Then the linear map $\mathcal{B}_{N_0} : \mathfrak{p} \to \mathfrak{p}$ taking P to $[P, N_0]$ is skew symmetric and its range is orthogonal to its kernel, given by the set of symmetric matrices commuting with N_0. If N_0 has simple spectrum, the description of the tangent space to the orbit then follows. The equation of motion for H_{20} is easy to compute, once one observes that only S may vary in an orbit in $\widetilde{\mathfrak{g}}_1^*$. □

Remark 3.4 (a) From the proof of Proposition 3.2, it should be clear that for $X \in \widetilde{\mathfrak{g}}_n^*$, the equations in (3.2) can be regarded as Hamiltonian systems on the coadjoint orbits of the Lie group LG_-^{Σ}, when the dual $(L\mathfrak{g}_-^{\sigma})^*$ of its Lie algebra is identified with the collection of power series $S_0 + N_1 z + \dots$, where the coefficients of the powers $z^k, k = 0, 1, \dots$ are alternatively symmetric and skew-symmetric real matrices. The advantage of using the slightly more general formulation here is that it puts the groups LG_+^{Σ} and LG_-^{Σ} on equal footing. Otherwise, the relevance of LG_+^{Σ} in the solution formula in Proposition 5.1 may look somewhat mysterious.

(b) The Bloch-Iserles equation can be regarded as an isospectral deformation of a general $n \times n$ matrix keeping the skew-symmetric part fixed. Indeed, if we set $z = 1$ in $(S + zN)^{\cdot} = [\,NS + SN + zN^2, S + zN\,]$ and let $M = S + N$, then a straightforward calculation shows that

$$\dot{M} = \frac{1}{4}\left[\,(M^T M + M M^T) + (M^T)^2, M\,\right]. \tag{3.7}$$

Conversely, if M satisfies the above equation, then

$$\dot{M^T} = \frac{1}{4}[\,(M^T M + M M^T) + M^2, M^T\,] \tag{3.8}$$

and by a direct computation, we find $(M - M^T)^{\cdot} = 0$. Now, the Bloch-Iserles equation preserves the spectral curve $\det(S + zN - w) = 0$. (See Remark 4.3 below.) On the other hand, the Toda flow in [DLT1] also has a spectral curve, given by $\det(M + h(M^T - M) - w) = 0$. If we write $M = S + N$, then by changing the variable

h above in the obvious way, we can make the Toda curve coincide with $\det(S + zN - w) = 0$. Percy Deift asked us if (3.7) might arise from a combination of flows associated to the Toda curve. To answer this question, note that any flow generated by a combination of the coefficients from the Toda curve must be of the form $\dot{M} = [\,\Pi_{\mathfrak{l}}\, F(M, M^T), M\,]$. (See [DL].) Here, $\mathfrak{l}$ is the Lie subalgebra of $\mathfrak{g}$ consisting of lower triangular matrices and $\Pi_{\mathfrak{l}}$ is the projection map to $\mathfrak{l}$ relative to the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$, and $F(M, M^T)$ is a polynomial in M and M^T. If we denote the right hand side of (3.7) by $X(M)$, then it is obvious that $X(OMO^T) = OX(M)O^T$ for any orthogonal matrix O. However, the expression $[\,\Pi_{\mathfrak{l}}\, F(M, M^T), M\,]$ clearly does not satisfy this invariance property. Hence the answer to the above question is in the negative. From a different point of view, it is also tempting to see if one can constrain the flows arising from the Toda curve to the submanifold $Q = \{\, M \in \mathfrak{g} \mid M - M^T = N \,\}$ where N is a fixed skew-symmetric matrix. Unfortunately, the submanifold Q is not a cosymplectic submanifold in the sense of Weinstein [W]. (The cosymplectic submanifolds are the generalization in a Poisson context where we can carry out Dirac's idea of constraining a Hamiltonian system.) Thus this idea also fails.

4. Liouville integrability.

Let $sym_{ij}(A, B)$, the *ij-symmetrizer* of matrices A and B, denote the sum of all monomials in A and B of degree $i + j$ consisting of i $A's$ and j $B's$.

Lemma 4.1. *(a)* $[\ sym_{i,j+1}(A, B)\ ,\ A\ ] + [\ sym_{i+1,j}(A, B)\ ,\ B\ ] = 0$.
(b) Let $A = S$ be symmetric and $B = N$ skew-symmetric. Then $sym_{ij}(S, N)$ is symmetric (resp. skew-symmetric) if j is even (resp. odd).
(c) If A and B are $n \times n$ matrices, than each symmetrizer $sym_{n-\ell,\ell}(A, B), \ell = 0, \dots, n$, is a linear combination of symmetrizers $sym_{r,s}(A, B)$, of smaller degree $r + s < n$.
(d) For generic choices of A and B (i.e., for an open, dense set of pairs (A, B)), $sym_{k-\ell,\ell}(A, B), k, \ell = 0, \dots, n - 1$, are linearly independent. The same is true for generic choices of $A = S$ symmetric and $B = N$ skew-symmetric.

Proof. To simplify notation, we omit the obvious matrix dependence of sym_{ij}. Statement (a) follows from

$$sym_{i+1,j+1} = A\ sym_{i,j+1} + B\ sym_{i+1,j} = sym_{i,j+1}\ A + sym_{i+1,j}\ B.$$

The first equality states that any monomial in $sym_{i+1,j+1}$ either starts with A and is completed with i $A's$ and $j + 1$ $B's$ or starts with B and is completed with $i + 1$ $A's$ and j $B's$. The second equality is obtained by taking into account the last matrix of each monomial. The proof of (b) is obvious. To see (c), notice that the claim for $i = 0$ (resp. $i = n$) is just the Cayley-Hamilton theorem for B (resp. A). More generally, write the Cayley-Hamilton theorem for the matrix $M = A + zB$,

$$-M^n = \sigma_1 M^{n-1} + \sigma_2 M^{n-2} + \ldots + \sigma_n M^0,$$

where $\sigma_j = \sigma_j(z)$, a standard symmetric function of the eigenvalues of M, is a polynomial of degree j in z. Now collect terms in z: on the left hand side, the coefficient of z^j is of the form $sym_{n-\ell,\ell}$; on the right hand side, it is a linear combination of symmetrizers of degree smaller than n. To prove (d), let B_* be the matrix whose only nonzero entries, equal to one, lie along the subdiagonal of entries with indices (r,s) satisfying $r-s=1$ and let A_* be a diagonal matrix with entries along the diagonal in geometric progression $c^1, c^2, \ldots, c^n$. Clearly, $sym_{k-\ell,\ell}$ only has nonzero entries along the subdiagonal $r-s=\ell$, forming a geometric progression of order $c^{k-\ell}$. Thus, to check if a linear combination of symmetrizers is independent, it suffices to consider the independence of subsets of symmetrizers consisting of a fixed index ℓ (keep in mind that, by hypothesis, $\ell < n$), for which $k-\ell$ runs from 0 to $n-\ell-1$. We thus have to prove the linear independence of $n-\ell$ vectors in $\mathbb{R}^{n-\ell}$, with coordinates in geometric progressions of ratio $c^1, \ldots, c^{n-\ell-1}$: this is clearly true for a generic choice of c (we thank Nicolau Saldanha for suggesting matrices A_* and B_*). From analyticity, independence holds for generic choices of A and B, proving the first part of (d). We now show that generic independence still holds for symmetric (resp. skew-symmetric) choices of A (resp. B). By continuity, independence still holds for matrices sufficiently close to A_* and B_*. In particular, this is still true if we keep $S_* = A_*$ and change B_* by adding a small negative number on its subdiagonal of entries (r,s) for which $r-s=-1$, giving rise to a matrix $\tilde{B_*}$. Now, consider a diagonal matrix D so that $N_* = D\tilde{B}_*D^{-1}$ is skew-symmetric. The pair $S_* = DA_*D^{-1}$ and N_* also has independent symmetrizers of degree less than n and genericity for pairs (S,N) follows again by analiticity. □

As usual, let S and N be symmetric and skew-symmetric $n\times n$ matrices. The set of symmetrizers $sym_{k-\ell,\ell}(S,N)$ having degree $k<n$ which are symmetric matrices (i.e., those for which j is even, by the previous lemma) splits in two sets. The first will give rise to the family of commuting integrals $H_{k\ell}$, the second relates to the Casimirs of generic coadjoint orbits within $\widetilde{\mathfrak{g}}_1^*$:

$$\begin{aligned}\mathcal{I} &= \{sym_{k-\ell,\ell},\ 1\le k\le n-1,\ 0\le \ell\le n-2,\ \ell \text{ even}\},\\ \mathcal{S}_N &= \{sym_{\ell\ell},\ 0\le\ell\le n-1,\ \ell\text{ even}\}.\end{aligned}$$

From Lemma 4.1, for a generic choice of S and N, the map $\mathcal{B}_N$ is injective when restricted to the vector space spanned by $\mathcal{I}$: in particular, the matrices $[\ sym_{k-\ell,\ell}(S,N)\ ,\ N\ ]$, for $sym_{k-\ell,\ell}\in\mathcal{I}$, are linearly independent.

Theorem 4.2. *(a) Let N_0 have simple spectrum and consider an element $S_0 + zN_0 \in \widetilde{\mathfrak{g}}_1^*$. Then $\mathcal{O}_{S_0+zN_0}$ is a $2[n^2/4]$ dimensional affine linear space given by*

$$\left\{(S+zN)\in\widetilde{\mathfrak{g}}_1^* \mid N=N_0,\, tr(SN^\ell)=tr(S_0N_0^\ell), \ell \text{ even } , 0\le\ell\le n-1\right\}. \quad (4.2)$$

(b) The equation of motion of the Hamiltonian $H_{k\ell}$ is given by

$$\dot{S} = [\ sym_{k-(\ell+1),\ell+1}(S,N)\ ,\ S\ ] = -[\ sym_{k-\ell,\ell}(S,N)\ ,\ N\ ],\quad \dot{N}=0.$$

The Hamiltonians in the set

$$\{H_{k\ell},\ 1 \le k \le n-1,\ 0 \le \ell \le n-2,\ \ell\ \textit{even}\}$$

are generically independent on $\mathcal{O}_{S_0+zN_0}$ *and provide* $[n^2/4]$ *commuting integrals.*

Proof. Statement (a) follows from the characterization of the tangent space to the orbit given in Proposition 3.3, combined with the independence of the elements of $\mathcal{C}$. The first formula for the equation of motion for $H_{k\ell}$ follows by expanding $X(z)^k$ in the equation of motion as given in Proposition 3.1. The second formula is a consequence of Lemma 4.1(a). Finally, by considering the second description, generic independence of the vector fields is a consequence of the remarks just above the statement of the theorem. □

Remark 4.3 Equivalently, we can consider the conserved quantities given by the coefficients of the characteristic polynomial $p(z,w) = det(S + zN - wI)$. Notice that $(S + zN)^T = S - zN$, so $p(z,w)$ is an even function of z :

$$p(z,w) = det(S + zN - wI) = \sum_{r=0}^{N}\sum_{k=0}^{[\frac{r}{2}]} I_{rk}(S,N) z^{2k} w^{N-r}.$$

The fact that the functions I_{rk} Poisson commute also follows by coadjoint invariance. Generic independence follows from the generic independence of the $H_{k\ell}$. Note that the functions $I_{2k,k}$, $k = 1, \ldots, [n/2]$ clearly depends only on N and therefore are trivial integrals. On the other hand, it is easy to see that $\{I_{2k+1,k}\}_{0\le k\le[\frac{n+1}{2}]-1}$ is equivalent to the coadjoint orbit invariants $\{tr(SN^{2k})\}_{0\le k\le[\frac{n-1}{2}]}$. Hence the nontrivial integrals are given by $\{I_{rk}\}_{0\le k\le[r/2]-1, 1\le r\le n}$ and this also gives a total of

$$\sum_{r=1}^{n}\left[\frac{r}{2}\right] = \left[\frac{n^2}{4}\right]$$

conserved quantities, as required. The fact that the *spectral curve* $p(z,w) = 0$ is preserved by the Bloch-Iserles equation means that the corresponding flow is linearized on the corresponding Jacobian variety and that the solution can be explicitly written down in terms of Riemann theta functions. We shall leave the details to the interested reader. (See, however, Proposition 5.2 and Remark 5.3 in this connection.)

5. Solution by factorization.

The flows associated to the integrals $H_{k\ell}$ may be described by an explicit formula. Let $\mathbb{D}_+, \mathbb{S}^1$ and $\mathbb{D}_-$ be respectively, the sets of complex numbers z together with $z = \infty$ for which $|z| \le 1$, $|z| = 1$ and $|z| \ge 1$. Recall that ([PS]) a loop $\gamma \in LG$ admits a *Birkhoff factorization* $\gamma(z) = \gamma_+(z)\ d(z)\ \gamma_-^{-1}(z)$. Here, γ_+ and γ_- are

restrictions to $\mathbb{S}^1$ of analytic functions extending to the boundary of $\mathbb{D}_+$ and $\mathbb{D}_-$ taking their values on $GL(N,\mathbb{C})$, the matrix $d(z)$ is diagonal with diagonal entries of the form $z^{a_i}, a_i \in \mathbb{Z}$, and $\gamma_-(\infty) = I$, the identity matrix. The only ingredient which is not automatic in the derivation of the formula for the flows is the proof of the triviality of the diagonal factor in the factorization of the loops associated to the initial conditions.

As stated, the Birkhoff factorization of an invertible loop is not necessarily unique. What is true, and follows from an argument similar to the given in the proof of the proposition below, is that different factorizations have the same diagonal factor, up to permutation of its diagonal entries. The loops we have to consider, however, admit a special symmetry. We follow in spirit the arguments in [GK].

Proposition 5.1. *Let γ be a loop satisfying the symmetry $\gamma(z)\gamma^T(-z) = I$. Then its Birkhoff factorization has trivial diagonal factor, $d(z) = I$, and is unique. Moreover, the nontrivial factors also satisfy the symmetry, $\gamma_\pm(z)\gamma_\pm^T(-z) = I$.*

Proof. Suppose $\gamma(z) = \gamma_+(z)d(z)\gamma_-^{-1}(z)$, with the notation above: we have to show that all the integers a_i are equal to 0. For a continuous function $f : \mathbb{S}^1 \to \mathbb{C}^*$, define its winding number $w(f)$ to be the (signed) number of turns of its image around the origin. The winding number is invariant under continuous deformations f_t through functions which avoid the origin and is additive with respect to products, $w(fg) = w(f) + w(g)$. Since γ_+ (resp. γ_-) extends to $\mathbb{D}_+$ (resp. $\mathbb{D}_-$), taking values at invertible matrices, one may deform the loop γ to a constant loop through loops taking values on invertible matrices, so that $w(\det\gamma_+) = w(\det\gamma_-) = 0$. By additivity, $w(\det\gamma) = w(\det\, d) = \sum_i a_i$. On the other hand, since $\gamma(z)\gamma^T(-z) = I$ and $w(\det\gamma(z)) = w(\det\gamma^T(-z))$, we must have $w(\det\gamma) = 0$. Thus, $\sum_i a_i = 0$. Also from $\gamma(z)\gamma^T(-z) = I$, we have

$$d(z)\gamma_-^{-1}(z)\gamma_-^{-T}(-z)d(-z) = \gamma_+^{-1}(z)\gamma_+^{-T}(-z).$$

Equating diagonal entries of both sides, we obtain, for $i = 1, \dots, n$,

$$(\gamma_-^{-1}(z)\gamma_-^{-T}(-z))_{ii} = (-1)^{a_i} z^{-2a_i}(\gamma_+^{-1}(z)\gamma_+^{-T}(-z))_{i,i},$$

which we denote, with the obvious attributions, by $f_-(z) = z^{-2a_i}g_+(z)$. Clearly, the hypothesis $\gamma_-(\infty) = I$ implies $f_-(\infty) = 1$. Suppose $a_i < 0$ for some i. Then the function $h(z)$, which agrees with f_- on $\mathbb{D}_-$ and with $z^{-2a_i}f_+(z)$ on $\mathbb{D}_+$ is an entire, bounded function — a constant — satisfying $h(0) = 0$ and $h(\infty) = 1$. The upshot is that $a_i \geq 0$, and since $\sum_i a_i = 0$, we must have $a_i = 0$ for all i. A similar argument involving Liouville's theorem obtains uniqueness of factorization. Symmetry of $\gamma_\pm$ in turn follows from unique factorization applied to the equation $\gamma(z) = \gamma_+(z)\gamma_-^{-1}(z) = \gamma(-z)^{-T} = \gamma_+(-z)^{-T}\gamma_-(-z)^T$. □

As we noted earlier, the vector space $L\mathfrak{g}^\sigma$ equipped with the Lie bracket in (2.11) coincides with $\widetilde{\mathfrak{g}}$. Since the modified Yang-Baxter equation (eqn. (2.10)) is a factorization condition, our next result is standard ([STS]). We sketch the argument in order to check that the symmetries required in the previous proposition to obtain the Birkhoff factorization with trivial diagonal factor indeed hold.

Proposition 5.2. *Let $f_{k\ell}(x,z) = x^k z^{-(\ell+1)}$, for ℓ even. Then the solution of the equation of motion*

$$\dot{X}(z) = \left[\,\Pi_+((X(z)^k z^{-(\ell+1)}), X(z)\,\right] = \left[\,\Pi_+ f_{k\ell}(X(z),z), X(z)\,\right], \quad X(0,z) = X_0(z),$$

generated by the Hamiltonian $H_{k\ell}$ is given by

$$X(t,z) = g_-^{-1}(t,z)X_0(z)g_-(t,z) = g_+^{-1}(t,z)X_0(z)g_+(t,z),$$

where g_+ and g_- are obtained from the Birkhoff factorization

$$exp(-tf_{kl}(X_0(z),z)) = g_+(t,z)g_-^{-1}(t,z).$$

Proof. First, notice that, for any real t, the loop $\gamma(t,z) = exp(-tf_{k\ell}(X_0(z),z))$ satisfies the hypothesis of the previous proposition. Indeed, it suffices to check that, for even ℓ, $\delta(z) = -f_{k\ell}(X_0(z),z)$ satisfies the linearization $\delta(z)+\delta^T(-z) = 0$, which is obvious. Thus g_- and g_+ are uniquely determined from $exp(-tf_{k\ell}(X_0(z),z)) = g_+(t,z)g_-^{-1}(t,z)$. Differentiating the above expression, we obtain

$$\begin{aligned} &exp(-tf_{k\ell}(X_0(z),z))\ f_{k\ell}(X_0(z),z) \\ &= \dot{g}_+(t,z)g_-^{-1}(t,z) - g_+(t,z)g_-^{-1}(t,z)\dot{g}_-(t,z)g_-^{-1}(t,z) \end{aligned}$$

so that

$$g_+^{-1}(t,z)\dot{g}_+(t,z) = -\Pi_+(g_+^{-1}(t,z)f_{k\ell}(X_0(z),z)g_+(t,z)) = -\Pi_+ f_{k\ell}(X(t,z),z)$$

for $X(t,z) = g_+^{-1}(t,z)X_0(z)g_+(t,z) = g_-^{-1}(t,z)X_0(z)g_-(t,z)$. Taking the derivative of the first expression for $X(t,z)$, we obtain

$$\dot{X}(t,z) = \left[\,-g_+^{-1}(t,z)\dot{g}_+(t,z), X(t,z)\,\right] = \left[\,\Pi_+ f_{k\ell}(X(t,z),z)), X(t,z)\,\right].$$

□

Remark 5.3 The Birkhoff factorization for $g_\pm(t,z)$ above can be solved explicitly in terms of Riemann theta functions. See, for example, [RS] and [DL] for details.

Remark 5.4 In the nonperiodic Toda hierarchy, there are vector fields whose solution for a given initial condition S_0 are especially simple at integer times. One example is associated to the QR factorization of $exp(tf(S_0))$ for $f(x) = \ln(x)$. This evolution is related to the QR iteration, a starting point of many algorithms to compute eigenvalues ([S],[DNT]). In a similar fashion, there are evolutions related to some of the Hamiltonians $H_{k\ell}$ which are algebraically solvable at integer times, since the Birkhoff factorization may be performed explicitly on loops of matrices with polynomial entries.

6. A finite dimensional group for the Bloch-Iserles equation.

We now indicate how to obtain a finite dimensional group which induces a coadjoint orbit diffeomorphic to $\mathcal{O}_{S_0+zN_0}$, on which the Bloch-Iserles equation arises as a Hamiltonian system. This construction answers a question addressed in [BI].

Represent a loop $g(z) = (I + \frac{g_{-1}}{z} + \frac{g_{-2}}{z^2} + \dots) \in LG_-^\Sigma$ (set $g_0 = I$) as the bi-infinite (convolution) matrix M_g which, in Fourier variables, corresponds to multiplying a (matrix) function defined on the circle by $g(z)$. The matrix M_g is obtained by prescribing the same basis $\{\dots, z^{-2}, z^{-1}, z^0, z, z^2, \dots\}$ on domain and range. It is block upper triangular, with $n \times n$ blocks indexed by $(M_g)_{ij}$, equal to g_{-k} for $i - j = -k$. Upper unipotent convolution matrices form a group $\mathcal{M}$, and matrices associated to loops admitting the symmetry $g(z)g^T(-z) = I$ form a subgroup $\mathcal{M}^\Sigma$. Consider the normal subgroup $\mathcal{M}_3^\Sigma$ of $\mathcal{M}^\Sigma$ on which the diagonals associated to $k = 1$ and 2 have zero entries: such matrices correspond to loops of the form $I + O(z^{-3})$. The quotient $\mathcal{M}^\Sigma/\mathcal{M}_3^\Sigma$ corresponds to 'forgetting' diagonals associated to $k \geq 3$ and is clearly isomorphic to the group $G_f = LG_-^\Sigma/LG_{-3}^\Sigma$ defined in the Introduction. Elements in the quotient may be represented by matrices M_g^0, on which diagonals for which $k \geq 3$ are set equal to zero. Now, set Π_3 denote the orthogonal projection on the three basis elements $1, z, z^2$. A simple computation shows that $\mathcal{M}^\Sigma/\mathcal{M}_3^\Sigma$ is (group) isomorphic to the $3n \times 3n$ matrices of the form $\Pi_3 M_g^0 \Pi_3^*$ — this is the finite group which will induce the Bloch-Iserles equation, as we shall see.

We now provide the concrete description of $\mathcal{M}^\Sigma/\mathcal{M}_3^\Sigma$. Consider the group of real $3n \times 3n$ matrices of the form

$$g = g(S,N) = \begin{pmatrix} I & S & \frac{S^2}{2} + N \\ 0 & I & S \\ 0 & 0 & I \end{pmatrix},$$

where the entries are $n \times n$ matrices. Here, matrices $S, T, \tilde{S}, \tilde{T}$ are symmetric, $N, M, \tilde{N}, \tilde{M}$ are skew symmetric. A simple computation gives $g(S,N)g(T,M) = g(S + T, N + M + \frac{[S,T]}{2})$, in agreement with the loop product

$$(I + \frac{S}{z} + \frac{\frac{S^2}{2} + N}{z^2} + \dots)(I + \frac{T}{z} + \frac{\frac{T^2}{2} + M}{z^2} + \dots) = \\ I + \frac{S+T}{z} + \frac{\frac{(S+T)^2}{2} + N + M + \frac{[S,T]}{2}}{z^2} + \dots .$$

The Lie algebra $\mathfrak{g}_f$ and its dual $\mathfrak{g}_f^*$ consist of elements of the form

$$X(S,N) = \begin{pmatrix} 0 & S & N \\ 0 & 0 & S \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A(S,N) = \begin{pmatrix} 0 & 0 & 0 \\ \tilde{S} & 0 & 0 \\ 2\tilde{N} & \tilde{S} & 0 \end{pmatrix}.$$

Here the nondegenerate pairing is $(X, A) = trXA$ and the usual computations yield $Ad^*_{g(T,M)}(A(S,N)) = A(S + [N,T], N)$. Up to trivial identifications, this is exactly

the formula for the coadjoint action of LG^{Σ}_{-} through the loop $S + zN$ described in Section 3. Clearly, the same Hamiltonians defined in Section 3 induce the same flows in this setup.

7. Another coadjoint orbit.

From formula (2.18), it is clear that an $n \times n$ matrix N_0 with a large kernel gives rise to especially small coadjoint orbits. or a concrete example, split an arbitrary symmetric matrix S into blocks,

$$S = \begin{pmatrix} a & b & u^T \\ b & c & v^T \\ u & v & B \end{pmatrix} \tag{6.1}$$

where a, b and c are real numbers, u and v are vectors of dimension $n-2$ and B is a real, symmetric matrix of dimension $n-2$. A direct computation shows that the flow $\dot{S} = [N, S^2]$ is equivalent to the equations

$$\begin{aligned} \dot{a} &= 2\langle u, v\rangle, \\ \dot{b} &= \langle v, v\rangle - \langle u, u\rangle, \\ \dot{c} &= -2\langle u, v\rangle = -\dot{a}, \\ \dot{B} &= 0. \end{aligned} \tag{6.2}$$

With the same notation as above, the equation $\dot{S} = [N, S^3]$ reads

$$\begin{aligned} \dot{a} &= 2b^3 + 2ba^2 + 2b(\langle v, v\rangle + \langle u, u\rangle) + 2\langle u, Bv\rangle, \\ \dot{b} &= -2a^3 - 2ab^2 - -2a(\langle u, u\rangle + \langle v, v\rangle) + \langle v, Bv\rangle - \langle u, Bu\rangle, \\ \dot{c} &= -\dot{a}, \\ \dot{u} &= (b^2 + a^2)v + bBu - aBv + \langle u, v\rangle u + \langle v, v\rangle v + B^2 v, \\ \dot{v} &= -(b^2 + a^2)u - aBu - bBv - \langle u, u\rangle u - \langle u, v\rangle v - B^2 u. \end{aligned} \tag{6.3}$$

We select a simple special case. Suppose u and v are points in a possibly complex vector space V. Say B is a symmetric matrix (i.e., $B^T = B$) acting on V, so that its entries may be complex, and suppose that V splits in two subspaces V_e and V_o (we will denote them by even and odd vectors), which are real orthogonal to each other and are interchanged by B. In particular, we have $\langle u, B^{2k+1}v\rangle = 0$, for all natural k, for u and v with the same parity. Finally, suppose that, at time 0, the entries a, b and c are 0, and the vectors u and v have the same parity. Then we must have that a, b and c are kept constant equal to 0 and u and v preserve their parity. Indeed, the requirement that a, b and $c = -a$ are constant equal to zero is compatible with their evolutions, and the remaining equations, keeping the notation above to denote the real inner product on V, become

$$\begin{aligned} \dot{u} &= \langle u, v\rangle u + \langle v, v\rangle v + B^2 v, \\ \dot{v} &= \langle u, u\rangle u - \langle u, v\rangle v - B^2 u, \end{aligned} \tag{6.4}$$

which respect parity. From uniqueness of solutions for a differential equation, whatever solves this smaller system actually is the unique solution of the original system provided a, b and c stay put.

We consider an infinite dimensional version of the above system. Take $B = iD_x$, so that $B^2 = -D_{xx}$, and take $u(x)$ and $v(x)$ to be simultaneously either even or odd functions in the variable x. Let the inner product denote the usual real product in $L^2(dx)$. The evolution equations for the functions $u(t,x)$ and $v(t,x)$ read

$$\begin{aligned} u_t &= \langle u, v\rangle u + \langle v, v\rangle v - v_{xx}, \\ v_t &= \langle u, u\rangle u - \langle u, v\rangle v + u_{xx}. \end{aligned} \tag{6.5}$$

In particular, this integro-differential system preserves the original parity and the reality of the initial conditions.

Other partial differential equations which fit the format $\dot{S}(t) = [N, S^k]$ may be obtained as follows. Set $S = M_q$ the operator given by multiplication by the function $q(x)$. Choose now $N = DM_\alpha + M_\alpha D$, where D is the partial derivative with respect to x. Clearly, N is skew-symmetric and the evolution equation for the operators becomes $q_t = 2\alpha(q^k)_x$. These differential equations admit hierarchies of infinitely many conserved quantities, but some of these conserved quantities generate vector fields which are not differential equations: in a sense, the coadjoint orbit through the initial condition is too large.

References

[BI] Bloch, A. and Iserles, A., *On an isospectral Lie-Poisson system and its Lie algebra*, Preprint (2005).

[DL] Deift, P. and Li, L.-C., *Generalized affine Lie algebras and the solution of a class of flows associated with the QR eigenvalue algorithm*, Comm. Pure Appl. Math. **42** (1989), 963-991.

[DLT1] Deift, P., Li, L.-C. and Tomei, C., *Matrix factorizations and integrable systems*, Comm. Pure Appl. Math. **42** (1989), 443-521.

[DLT2] Deift, P., Li, L.-C. and Tomei, C., *Loop groups, discrete versions of some classical integrable systems, and rank 2 extensions*, Mem. Amer. Math. Soc. **2** (1992), no. 479.

[DNT] Deift, P., Nanda, T. and Tomei, C., *Differential equations for the symmetric eigenvalue problem*, SIAM J. Num. Anal. **20** (1983), 1-22.

[F] Flaschka, H., *The Toda lattice, I*, Phys. Rev. B **9** (1974), 1924-1925.

[GK] Gohberg, I. and Krein, M., *Systems of integral equations on a half-line with kernels depending on the difference of the arguments*, AMS Translations **14** (1960), 217-287.

[PS] Pressley, A. and G. Segal, *Loop groups*, Oxford Mathematical Monographs, Oxford University Press, 1986.

[Mo] Moser, J., *Finitely many mass points on the line under the influence of an exponential potential — an integrable system*, Lecture Notes in Mathematics **38** (1975), 467-497.

[Ma] Manakov, S.V., *Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body*, Funct. Anal. and its Appl. **10** (1976), 328-329.

[R] Reiman, A., *Integrable Hamiltonian systems connected with graded Lie algebras*, J. Soviet Math. **19** (1982), 1507-1545.

[RS] Reyman, A. and Semenov-Tian-Shansky, M., *Reduction of Hamiltonian systems, affine Lie algebras and Lax equations II*, Invent. Math. **63** (1981), 423-432.

[S] Symes, W., *The QR algorithm and scattering for the finite nonperiodic Toda lattice*, Physica 4D (1982), 275-280.

[STS] Semenov-Tian-Shansky, M., *What is a classical r-matrix?*, Funct. Anal. Appl. **17** (1983), 259-272.

[W] Weinstein, A., *The local structure of Poisson manifolds*, J. Diff. Geom. **18** (1983), 523-557.

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